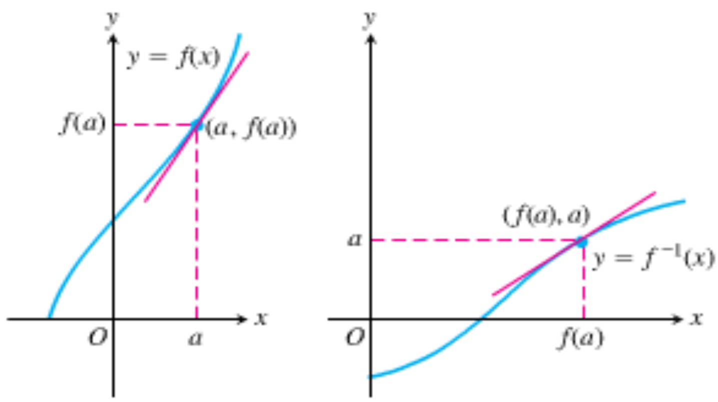
Derivatives of Inverse Functions



The slopes are reciprocal: $\left.\dfrac{df^{-1}}{dx}\right|_{f(a)} = \dfrac{1}{\left.\dfrac{df}{dx}\right|_{a}}$

Figure 3.52 The graphs of a function and its inverse. Notice that the tangent lines have reciprocal slopes.

THEOREM 3 Derivatives of Inverse Functions

If f is differentiable at every point of an interval I and df/dx is never zero on I, then f has an inverse and f^{-1} is differentiable at every point of the interval $f(I)$.

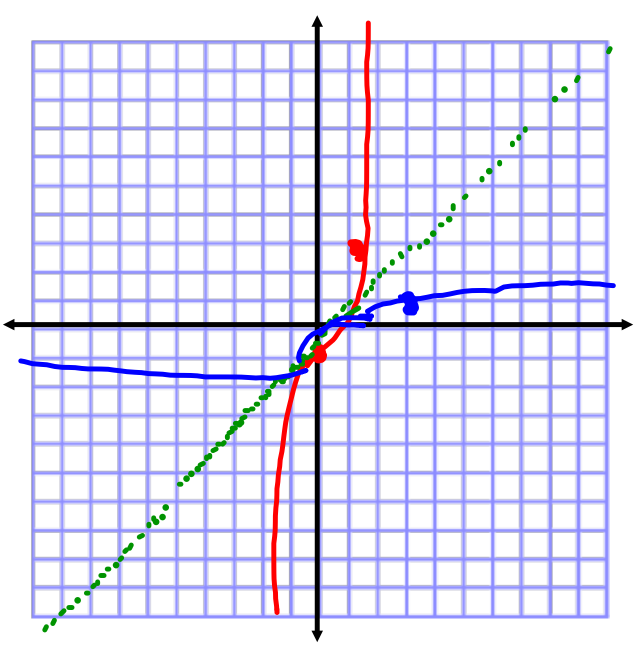
EXPLORATION 1 **Finding a Derivative on an Inverse Graph Geometrically**

Let $f(x) = x^5 + 2x - 1$. Since the point (1, 2) is on the graph of f, it follows that the point (2, 1) is on the graph of f^{-1}. Can you find

$f'(x) = 5x^4 + 2$ $$\frac{df^{-1}}{dx}(2),$$ $f'(1) = 7$

the value of df^{-1}/dx at 2, without knowing a formula for f^{-1}? $= \frac{1}{7}$

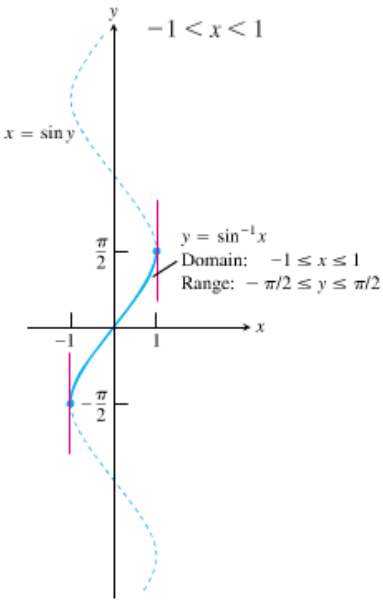
1. Graph $f(x) = x^5 + 2x - 1$. A function must be one-to-one to have an inverse function. Is this function one-to-one?
2. Find $f'(x)$. How could this derivative help you to conclude that f has an inverse?
3. Reflect the graph of f across the line $y = x$ to obtain a graph of f^{-1}.
4. Sketch the tangent line to the graph of f^{-1} at the point (2, 1). Call it L.
5. Reflect the line L across the line $y = x$. At what point is the reflection of L tangent to the graph of f?
6. What is the slope of the reflection of L?
7. What is the slope of L?
8. What is $\frac{df^{-1}}{dx}(2)$?



Derivative of the Arcsine

$$y = \sin^{-1} x$$

$-1 < x < 1$



Inverse Trig Function Derivatives Activity

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}.$$

If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule to get

$$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1 - u^2}}\frac{du}{dx}, \qquad |u| < 1.$$

$y = \sin x$

$\downarrow$ not equal

1. $x = \sin y \longrightarrow y = \sin^{-1} x$

2. $1 = \cos y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\cos y}$

3. $\sin^2 y + \cos^2 y = 1$

$\cos^2 y = 1 - \sin^2 y$

$\cos y = \sqrt{1 - \sin^2 y}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\sin y)^2}}$

4. $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$

1. $x = \tan y$

$1 = \sec^2 y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\sec^2 y}$

$\tan^2 y + 1 = \sec^2 y$

$= \frac{1}{\tan^2 y + 1}$

$= \frac{1}{x^2 + 1}$

$x = \sec y$

$1 = \sec y \tan y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$

$= \frac{1}{|x|\sqrt{x^2 - 1}}$

$\sec^2 y = \tan^2 y + 1$

$\tan^2 y = \sec^2 y - 1$

$\tan y = \pm\sqrt{\sec^2 y - 1}$

EXAMPLE 1 Applying the Formula

$$\frac{d}{dx}(\sin^{-1} x^2)$$

$$= \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$$

$$= \frac{2x}{\sqrt{1-x^4}}$$

EXAMPLE 2 **A Moving Particle**

A particle moves along the x-axis so that its position at any time $t \geq 0$ is $x(t) = \tan^{-1}\sqrt{t}$. What is the velocity of the particle when $t = 16$?

$$x(t) = \tan^{-1} t^{1/2}$$

$$v(t) = x'(t) = \frac{1}{1+(t^{1/2})^2} \cdot \frac{1}{2}t^{-1/2}$$

$$= \frac{1}{1+t} \cdot \frac{1}{2} t^{-1/2}$$

$$= \frac{1}{17} \cdot \frac{1}{2}\left(\frac{1}{4}\right)$$

$$= \frac{1}{136}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$$

EXAMPLE 3 Using the Form

$$\frac{d}{dx}\sec^{-1}(5x^4) =$$

$$= \frac{1}{5x^4\sqrt{(5x^4)^2-1}} \cdot 20x^3$$

$$= \frac{20x^3}{5x^4\sqrt{25x^8-1}}$$

$$= \frac{4}{x\sqrt{25x^8-1}}$$

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$

$$\cot^{-1} x = \pi/2 - \tan^{-1} x \qquad -\frac{1}{x^2+1}$$

$$\csc^{-1} x = \pi/2 - \sec^{-1} x \qquad -\frac{1}{|x|\sqrt{x^2-1}}$$

Use these identities to come up with the derivatives of these 3 inverse functions.

$$\frac{d}{dx} \cos^{-1} x = 0 - \frac{1}{\sqrt{1-x^2}}$$

$$= -\frac{1}{\sqrt{1-x^2}}$$

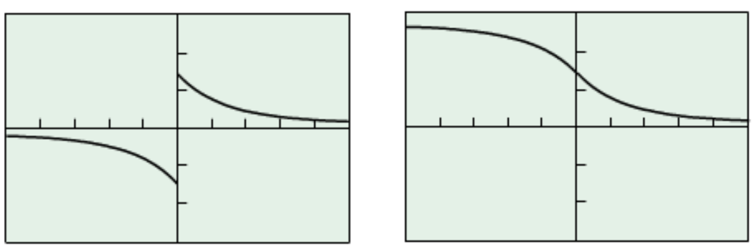
Calculator Conversion Identities

$$\sec^{-1} x = \cos^{-1}(1/x)$$

$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$

$$\csc^{-1} x = \sin^{-1}(1/x)$$

Notice that we do not use $\tan^{-1}(1/x)$ as an identity for $\cot^{-1} x$. A glance at the graphs of $y = \tan^{-1}(1/x)$ and $y = \pi/2 - \tan^{-1} x$ reveals the problem (Figure 3.55).



[–5, 5] by [–3, 3]
(a)

[–5, 5] by [–3, 3]
(b)

Figure 3.55 The graphs of (a) $y = \tan^{-1}(1/x)$ and (b) $y = \pi/2 - \tan^{-1} x$. The graph in (b) is the same as the graph of $y = \cot^{-1} x$.

EXAMPLE 4 A Tangent Line to the Arccotangent Curve

Find an equation for the line tangent to the graph of $y = \cot^{-1} x$ at $x = -1$.

$\left(-1, \frac{3\pi}{4}\right)$

$$y = \cot^{-1} x$$

$$y' = -\frac{1}{x^2+1}$$

$$= \frac{-1}{(-1)^2+1} = \frac{-1}{2}$$

$$y - \frac{3\pi}{4} = -\frac{1}{2}(x+1)$$